PARITY PATTERNS ON EVEN SEMIGRAPHS

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By a semigraph is meant an elementary cycle in the plane, completely triangulated on one side, and with no points or lines on the other side. The triangulated side may or may not contain points in addition to points of the cycle. If all points not points of the cycle are of even degree, the semi-graph will be called even. If there are no points other than points of the cycle, the semigraph is vacuously even. The graphs in Fig. 1 are

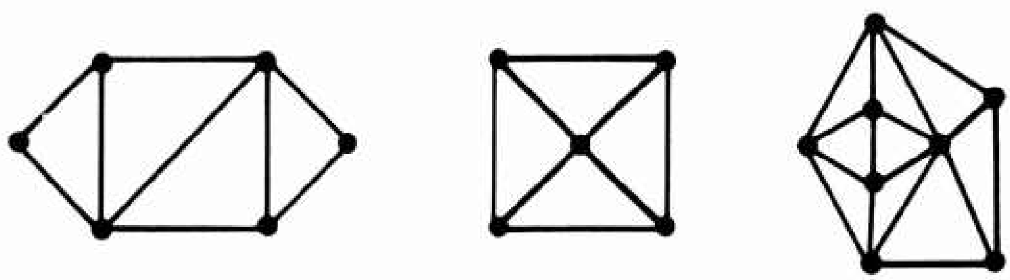

Fig. 1

all examples of even semi-graphs.

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For any point x of a graph, $p(x)$ (parity of x) is defined as

$$p(x) = e, \text{ degree of } x \text{ even,}$$
$$p(x) = o, \text{ degree of } x \text{ odd.}$$

For any cycle C, the corresponding cycle of $p(x)$'s, $x \in C$, will be called the parity-pattern $P(C)$ of the cycle.

Theorem: For any even semigraph G, with bounding cycle C and parity-pattern $P(C)$, there exists an even semigraph G' with bounding cycle C' of the same length as C with $P(C) = P(C')$ and G' contains at most one point not in C'.

Proof: By induction on the number of points n in C.

(1) The consequence holds for $n = 3$, since, for the triangle there is only one possible P. Either all points in C are even or two are odd, since the number of odd points must be even. But it can be shown that in a fully triangulated graph with all even nodes but two, these two cannot be adjacent.* Therefore the only P on C_3 is the case $p(x) = e$ for all points. This P is identical with that of a simple triangle.

(2) Assume the theorem holds for C_n. Consider a semigraph G with bounding cycle C_{n+1}.

Case 1. $n + 1$ is odd. In this case it is impossible that $p(x) = o$ for all $x \in C$. Hence, there is at least one $x \in C$, $p(x) = e$. Let y, z be the two neighbors of x in C. If there is a line between y and z, then y and z are the

*This theorem appears in Grunbaum, B., Convex Polytapes, to be published, Wiley, New York, 1966.

only neighbors of x. Construct G' by eliminating x. If there is no line between y and z, construct G' with bounding cycle C' by adding the line (y, z) to G (see Fig. 2).

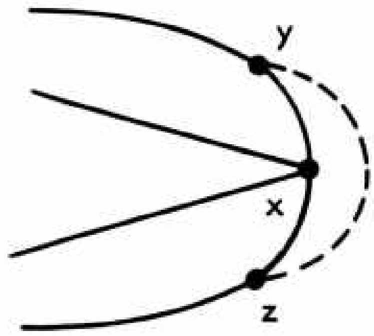

Fig.2

Either construction produces a new even semigraph with a bounding cycle of n points. By hypothesis, there is a semigraph G'' with a bounding cycle C'', P(C'') = P(C') and G'' contain at most one point not in C''. Now construct the graph G''' by adding the point x''' as in Fig. 3 to the points corresponding to y and z.

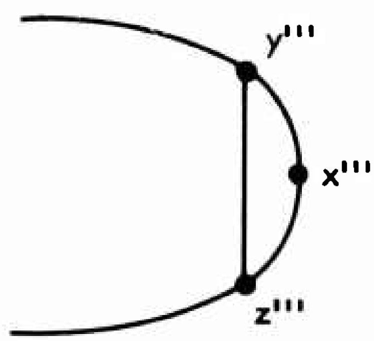

Fig.3

$p(x''') = p(x) = e$. In C'', the parities of y'' and z'' are reversed from the parities of the corresponding y and z in C, whence by adding the lines (y''', x''') (x''', z''') these parities are restored. Thus $P(C''') = P(C)$, and the theorem holds.

Case 2. $n + 1$ even. If $p(x) = o$ for every $x \in C$, G' consists of the simple cycle C' with the same number of points as C, with an additional point connected to each point of C'.

If $p(x) \neq o$ for some x, proof as in Case 1.